PMM U.S.S.R.,Vol.51,No.5,pp.640-644,1987
Printed in Great Britain

0021-8928/87 $10.00+0.00
© 1989 Pergamon Press plc

ON THE FORMATION OF A WAVE PACKET IN A BOUNDARY LAYER ON A FLAT PLATE*

E.D. TERENT'YEV

The framework of the theory of the boundary layer with selfinduced pressure /1, 2/ is used to study the initial stage of the development of the perturbations caused by triggering a vibrator. The values of the pressure are computed for short characteristic times, and this makes it possible to observe how a wave packet is formed.

The theoretical study of wave packets is usually restricted /3-6/ to quite large values of the characteristic time, measured from the instant of action on the boundary layer and such that the packet is practically already formed. A solution of the problem of the harmonic vibrator is constructed below for the short time intervals after the oscillator has been triggered and this, together with the results of /5, 6/, enable us to follow the development of linear perturbations in the boundary layer for any length of time.

Naturally, the theory of the boundary layer with selfinduced pressure is applicable, in its non-stationary version /7, 8/, only when the Reynolds numbers significantly exceed the critical value. It was shown in /9, 10/ that the linear version of the theory describes perturbations with wave numbers lying below the upper branch of the neutral stability curve, and the parameters corresponding to the largest growth increments refer to those described by the theory in question. In order to analyse the lengths and frequencies of the waves adjacent to the upper branch of the neutral stability curve, it is necessary to employ expansions /11-13/ different from those of the three-layer theory /1, 2/.

Let us consider the flow past a heat-insulated plate on which a harmonic vibrator is situated at a distance L^* from the leading edge. Let the incident flow be subsonic, with velocity $V_\infty{}^*$ directed along the plate, let $\rho_\infty{}^*$ be the density of the unperturbed gas and let $\lambda_{1\infty}{}^*$ be the first viscosity coefficient. We will introduce a small parameter $\varepsilon = R_1^{-1/8}$ $(R_1 = \rho_\infty{}^* V_\infty{}^* L^*/\lambda_{1\infty}{}^*)$ and choose the longitudinal dimensions of the vibrator $O(L^*\varepsilon^3)$, the oscillation amplitude $O(L^*\varepsilon^5)$ and the frequency $O(V_\infty{}^*/(L^*\varepsilon^2))$.

To describe the motion caused by such a vibrator, we can use the theory of the boundary layer with selfinduced pressure in its non-stationary version /7, 8/. When analysing the principal terms of the perturbations we find, that according to this theory the fundamental unperturbed motion of the gas is plane parallel. Using the dimensionless variables /8/ we shall specify the law of motion of the vibrator and its form, for the time $t > 0$

$$y_w = \sigma f(t, x) = \sigma f_1(x) \sin \omega_0 t, \quad \sigma \ll 1, \quad \omega_0 > 0$$

where ω_0 is the dimensionless frequency, x, y are the Cartesian coordinates, the function $f_1(x)$ defines a triangular form with parameters a and b ($f_1(x) = 0$ for $x \leqslant 0$, $2x$ for $0 < x \leqslant b$, $2b(a-x)/(a-b)$ for $b < x \leqslant a$, 0 for $x > a$). For the instants of time $t < 0$ we will put $y_w = 0$ and assume that the boundary layer is unperturbed.

The problem of a vibrator was studied in this formulation in /14/, where an expression for the pressure perturbation was written in terms of inverse Fourier and Laplace transforms. Making minor changes, we shall rewrite it in the form (l is a positive number)

$$p_1 = \pi^{-3/2} 2^{-1/2} \omega_0 \operatorname{Re}\left[\int_0^\infty k f_{1F}(k) e^{ikx}\, dk J(t, k, \omega_0)\right] \tag{1}$$

$$J = \int_{l-i\infty}^{l+i\infty} \Phi\, d\omega, \quad \Phi = \frac{\operatorname{Ai}'(\Omega) e^{\omega t}}{(\omega^2 + \omega_0^2) Q_3(\Omega, k)}, \quad \Omega = \frac{\omega}{(ik)^{2/3}}$$

$$f_{1F}(k) = -\sqrt{\frac{2}{\pi}} \frac{1}{k^2}\left(1 - \frac{a}{a-b} e^{-ikb} + \frac{b}{a-b} e^{-ika}\right)$$

$$Q_3(\Omega, k) = -\operatorname{Ai}'(\Omega) + i^{1/3} k^{1/3}\left[\frac{1}{3} - \int_0^\Omega \operatorname{Ai}(z)\, dz\right]$$

*Prikl.Matem.Mekhan.,51,5,814-819,1987

The analysis in /14, 15/ showed that when $t \to \infty$, expression (1) defines, in the region $1 \leqslant x < 2{,}4t$ with $\omega_0 > \omega_*$ ($\omega_* = 2{,}298$ is the frequency of neutral oscillations), a monochromatic Tollmien-Schlichting wave of frequency ω_0. The wave packet is distributed at distances $x > 2{,}4t$ /6/. A method of evaluating the integrals (1) given in /6/, enabled us to study the case when $t > 2{,}5$, but as the results obtained showed, the wave packet was already formed after such a time. The aim of this paper is to evaluate the integral (1) for $0 < t < 3$, i.e. for times at which the wave packet is being formed.

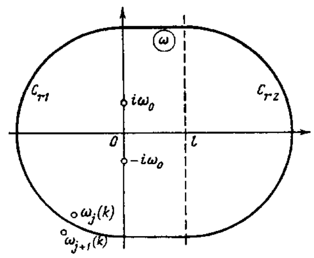

Fig.1

Numerical integration of the integral J involves large amounts of computer time due to the complicated nature of the integrand and the need to carry out the computations for a large number of values of k and various values of t. Therefore, we shall calcuate J using the methods of the theory of functions of a complex variable.

Let us consider the roots of the denominator of the integrand in the integral J. An analysis /13/ of the solutions of the dispersion equation $Q_3(\Omega, k) = 0$ has shown that k varies along the positive part of the real axis, and all its roots $\omega_n(k)$ are simple and lie to the left of the path of integration. When $k > 0$ is fixed and n increases. The relations $|\omega_n(k)| \to \infty$ and $\arg \omega_n(k) \to 4\pi/3$ hold. Let us draw on arc C_{r1} of radius r_0 so that it passes between the neighbouring roots $\omega_j(k)$ and $\omega_{j+1}(k)$ (Fig.1). Then by virtue of Cauchy's theorem we have

$$\int_{l-i\infty}^{l+i\infty} \Phi\, d\omega = 2\pi i\Big[\operatorname{res}\Phi(i\omega_0) + \operatorname{res}\Phi(-i\omega_0) + \sum_{n=1}^{j} \operatorname{res}\Phi(\omega_n(k))\Big] - \tag{2}$$

$$\int_{C_{r1}} \Phi\, d\omega, \quad \operatorname{res}\Phi(i\omega_0) = \frac{\exp(i\omega_0 t)\operatorname{Ai}'(\Omega_0)}{2i\omega_0 Q_3(\Omega_0, k)}$$

$$\operatorname{res}\Phi(\omega_n(k)) = \exp(\omega_n t)(ik)^{4/3}\operatorname{Ai}'(\Omega_n)[(\omega_n^2 + \omega_0^2)\times$$

$$(\omega_n + ik^2)\operatorname{Ai}(\Omega_n)]^{-1}, \quad \Omega_0 = \omega_0(ik)^{-2/3}, \quad \Omega_n = \omega_n(k)(ik)^{-2/3}$$

When r_0 increases, and therefore when j increases, the integral along the arc C_{r1} tends to zero. When $j \to \infty$ and k is fixed, the condition $|\omega_j| \sim j^{2/3}$ holds and the following estimate holds for the residues:

$$|\operatorname{res}\Phi(\omega_j)| \sim j^{-2/3}\exp(-a_0 t j^{2/3}), \quad a_0 > 0$$

Therefore when j increases, the series on the right-hand side of (2) will converge

$$J = \int_{l-i\infty}^{l+i\infty} \Phi\, d\omega = 2\pi i\Big[\operatorname{res}\Phi(i\omega_0) + \operatorname{res}\Phi(-i\omega_0) + \sum_{n=1}^{\infty} \operatorname{res}\Phi(\omega_n(k))\Big] \tag{3}$$

As the estimates in /14/ have shown, as $t \to \infty$, the first term of the series in n on the right-hand side of (3.3) will suffice. If, however, t is not large, then several terms of the series must be used. Calculations of the function $J(t, k, \omega_0)$ from (3), using the first 10 terms of the series, showed that when $t \geqslant 0{,}3$, the neglected terms become insignificant for all values of k except those in a small neighbourhood of $k = 0$. This neighbourhood, however, increases as t decreases, and an increasing number of terms has to be taken into account.

Let us construct a different method of calculating $J(t, k, \omega_0)$ for small values of time ($t \leqslant 0, 3$). To do this we consider a contour C_{r2} of radius r_0 situated in the right half-plane ω (Fig.1). Since the integrand Φ has no singularities in the finite part of the plane ω to the right of the contour of integration we have, according to Cauchy's theorem,

$$\int_{l-i\infty}^{l+i\infty} \Phi\, d\omega = -\lim_{r_0\to\infty} \int_{C_{r2}} \Phi\, d\omega$$

If the integrand tends exponentially to zero along the arc C_{r1} as $|\omega| \to \infty$, it increases exponentially along the arc C_{r2} as $|\omega|$ increases. At the same time, when k is fixed and $|\omega|$ increases, so does Ω and the inequality $-5\pi/6 \leqslant \arg\Omega \leqslant \pi/6$ holds for ω belonging to the arc C_{r2}. For such $|\Omega| \to \infty$ the derivative and the integral of the Airy function can be replaced by their asymptotic expansions /16/, and as a result we obtain

$$Q_3(\Omega, k)/\operatorname{Ai}'(\Omega) \sim -1 - i^{1/3}k^{4/3}\Omega^{-1}\sum_{n=0}^{\infty} C_{I,n}\Omega^{-3n/2} / \sum_{n=0}^{\infty} C_{d,n}\Omega^{-3n/2} \tag{4}$$

where $C_{I,n}$ are the coefficients of the expansion of $\int\limits_{\Omega}^{\infty} \mathrm{Ai}(z)\,dz$ and $C_{d,n}$ are the coefficients of the expansion of $\mathrm{Ai}'(\Omega)$.

Using a program for the division of polynomials, we shall use a digital computer to calculate the coefficients $C_{q,n}$ which yield the quotient from dividing two series on the right-hand side of (4). The coefficients $C_{q,n}$ will be real, since $C_{I,n}$ and $C_{d,n}$ are real. Expressing further Ω in terms of k and ω, dividing the polynomials in $\omega^{-1/2}$ once again and collecting in the quotient the terms of like order in k, we obtain a computer.

$$\Phi \sim \frac{e^{\omega t}}{\omega^2} \sum_{n=0}^{\infty} k^n \varphi_n(\omega_0, \omega), \qquad \varphi_n = \sum_{m=h(n)}^{g(n)} C_{m,n} \omega^{-m/2}$$

where $\varphi_n(\omega_0, \omega)$ are polynomials of degree $g(n)$ in the variable $\omega^{-1/2}$, with complex coefficients $C_{m,n}(\omega_0)$.

Integrating Φ with respect to the variable ω along the contour C_{r2} presents no difficulties, since standard methods can be used to show that

$$\int\limits_{C_{r2}} \frac{e^{\omega t}\,d\omega}{\omega^N} = \frac{2\pi t^{N-1}}{(N-1)!}, \qquad \int\limits_{C_{r2}} \frac{e^{\omega t}\,d\omega}{\omega^{N+1/2}} = \frac{2\sqrt{\pi}\,t^{N-1/2}}{(1/2)_N} \tag{5}$$
$$N = 1, 2, \ldots; \; (1/2)_N = 1/2\,(1/2 + 1)\ldots(1/2 + N - 1)$$

as $r_0 \to \infty$.

Therefore, we can write the integral J in the form of a series

$$J(t, k, \omega_0) = \sum_{n=0}^{\infty} C_n k^n \tag{6}$$

with complex coefficients C_n depending on ω_0 and t.

The problem of the convergence of the series on the right-hand side of (6) can be solved by making a sufficiently coarse estimate of the behaviour of the coefficients C_n as $n \to \infty$. Since the coefficients $C_{q,n}$ vary just as $C_{I,n}$, i.e. into proportion to $n!$ (the dependence on the powers of n can be disregarded) and the lower limit in the sum which gives φ_n can be assessed as $h(n) \sim 3n/2$, therefore in accordance with (5) we obtain

$$|C_n| \sim t^{3n/2} n!/[3n/2]! \tag{7}$$

The estimate (7) is rough, but yields important conclusions. Series (7) converges absolutely and has an infinite radius of convergence. When calculating J to the necessary degree of accuracy for given k, the increase in t results in the need to take an increasing number of terms into consideration. These properties are well illustrated by the calculation of the function J carried out for various values of k, t and ω_0. Since the series (6) converges absolutely, it can be considered in the whole complex k plane. Thus the series (6) defines an entire analytic function of the variable k, has no singularities in the finite part of J, and the point at infinity will represent the essential singularity of the series.

The construction of the representation of the function J with help of a series in residues (3) and in the form of a power series complement each other. Although both series converge absolutely and can be used for any values of k and t, nevertheless each series has its own domain of practical applicability connected with the fact that only a finite number of terms can be used in the computations. If we fix t, then several tens of terms of the series (6) will yield an expression for J with a high degree of accuracy for $k < k_b$, and conversely, several terms of the series (3) will give J with good accuracy when $k > k_b$. Finally, the limit value k_b is conditional. It depends on the variable t, the number of terms used in the series, and on the accuracy of the determination of J. If $t \leqslant 0{,}3$, then the boundary k_b will be sufficiently large and in order to compute the integrals from (1) in k, it will be sufficient to use expression (6), and in the case of $t \geqslant 1{,}5$ expression (3). The value $k_b = 2$ was chosen for the intermediate values of t.

When using a computer with series (6), we must utilize double accuracy (24 decimal places in the mantissa, ranging from -1232 to +1232). This is connected with the extraordinarily rapid growth of $C_{I,n}$ and $C_{d,n}$, and the very small values of the integrals (5). The computations were carried out using 150 terms of the series, and the highest value of k was $k = 7$.

Computing using series (3) were carried out using 10 terms, and the highest value of k was $k = 12$. Additional computations were carried out for $t \geqslant 2{,}5$ using only the first term $(n = 1)$ in the series (3), and the result agreed completely with that obtained using a 10-term approximation.

Fig.2 shows graphs of the pressure perturbations $p_1(t,x)$ generated by a triangular vibrator with parameters $a=2$; $b=1$, switched on at $t=0$ at the supercritical frequency $\omega_0=3{,}0$. When $t=0{,}1$ (Fig.2a) the function p_1 is almost symmetrical with respect to the vibrator shown in the figure with a dashed line. The perturbation reaches its highest absolute value above the centre of the vibrator. It becomes small towards its ends and takes the opposite sign at the ends, the value at the rear end of the vibrator slightly exceeding the value at the front end. A drop in pressure above the central part of the vibrator projected at the instant $t=0{,}1$ into the boundary layer is characteristic of external subsonic flow. When $t=0{,}2$ (Fig.2b), the pressure in the central part continues to fall as the vibrator rises, and the asymmetry intensifies due to the increase in pressure behind the rear end of the vibrator. When $t=0{,}4$, (Fig.2c), the pressure perturbation behind the rear end becomes

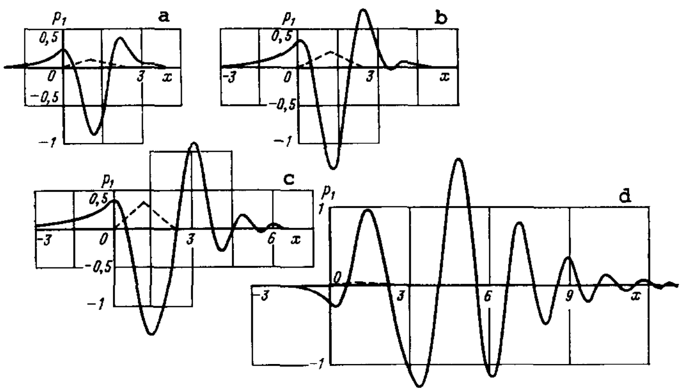

Fig.2

almost equal to that above the vibrator, and another complete oscillation can be fully discerned downstream. When $t=1{,}0$ (Fig.2d), the pressure perturbation takes a form characteristic of the wave packet whose development over a period of time yields the structure discussed in /6/. The initial stages of the development of perturbations in the boundary later described here, agree qualitatively with those observed experimentally*.(*Grek' G.R., Kozlov V.V. and RAMOZANOV M.P., Experimental study of the appearance and development of two-dimensional wave packets in a boundary layer. Preprint 11, Novosibirsk, In-t teoret. i prikl. mekhaniki SO AN SSSR, 1986.)

REFERENCES

1. NEILAND V.YA., On the theory of the detachment of a boundary layer in supersonic flow. Izv. Akad. Nauk SSSR, MZhG, 4, 1969.
2. STEWARTSON K., On the flow near the trailing edge of a flat plate. II, Mathematika, 16, 31, 1969.
3. GASTER M.A., Theoretical model of a wave packet in the boundary layer on a flat plate, Proc. Roy. Soc. A. 374, 1649, 1975.
4. RYZHOV O.S. and TERENT'YEV E.D., On certain properties of the vortex spots in a boundary layer on a plate. Dokl. Akad. Nauk SSSR, 276, 3, 1984.
5. RYZHOV O.S. and TERENT'YEV E.D., On a method of generating wave packets in a boundary layer. Modern Problems of the Mechanics of a Continuous Medium. Moscow, Izd-vo MFTI, 1985.
6. RYZHOV O.S. and TERENT'YEV E.D., On the transition mode characterizing the triggering of a vibrator in a subsonic boundary layer ona plate. PMM, 50, 6, 1986.
7. SCHNEIDER W., Upstream propagation of unsteeady disturbances in supersonic boundary layer, J. Fluid Mech. 63, 3, 1974.
8. RYZHOV O.S., Equations of a non-stationary boundary layer with selfinduced pressure. Dokl. Akad. Nauk SSSR, 234, 4, 1977.
9. SMITH F.T., On the non-parallel flow stability of the Blasius boundary layer, Proc. Roy. Soc. London. Ser. A. 366, 1724, 1979.
10. ZHUK V.I.and RYZHOV O.S., Free interaction and stability of a boundary layer in an incompressible fluid. Dokl. Akad. Nauk SSSR, 253, 6, 1980.
11. MIKHAILOV V.V., On the asymptotic form of the neutral curves of the linear problem of the stability of a laminar boundary layer. Izv. Akad. Nauk SSSR, MZhG, 5, 1981.

12. BODONYI R.J. and SMITH F.T., The upper branch stability of the Blasius boundary layer, including non-parallel flow effects, Proc. Roy. Soc. Lond. Ser. A. 375, 1760, 1981.
13. ZHUK V.I. and RYZHOV O.S., On the asymptotic form of the solutions of the Orr-Sommerfeld equations specifying unstable oscillations for large values of the Reynolds number. Dokl. Akad. Nauk SSSR, 268, 6, 1983.
14. TERENT'YEV E.D., The linear problem of a vibrator executing harmonic oscillations at supercritical frequencies in a subsonic boundary layer. PMM, 48, 2, 1984.
15. TERENT'EV E.D., Linear problem on vibrator on a flat plate in subsonic boundary layer, Laminar-turbulent transition IUTAM Symposium. Novosibirsk. 1984.
16. VAZOV V., Asymptotic Expansions of the Solutions of Ordinary Differential Equations. Moscow, Mir, 1968.

Translated by L.K.

PMM U.S.S.R.,Vol.51,No.5,pp.644-651,1987
Printed in Great Britain

0021-8928/87 $10.00+0.00
©1989 Pergamon Press plc

ASYMPTOTIC THEORY OF A WAVE PACKET IN A BOUNDARY LAYER ON A PLATE*

O.S. RYZHOV and I.V. SAVENKOV

The propagation of a wave packet generated by a point source in a boundary layer on a flat plate is considered. The fluid is assumed to be incompressible, and the distance from the leading edge of the plate is chosen to be so large, that the Reynolds number can be assumed to tend to infinity. The field of perturbed motion is constructed using the framework of the linearized theory of the boundary layer with selfinduced pressure, with help of expansions in Laplace integrals with respect to time and Fourier integrals with respect to two spatial variables. The saddle-point method is used to calculate the inverse transforms.

The pulsating motion of the fluid in the wave packet (laminar vortex spot) is characterized by a continuous frequency spectrum. The other special property of the wave packet is that the oscillations are modulated already in the linear stage of its propagation, and thanks to this the amplitude has a sharp maximum at the centre of the perturbed region. The mutual interaction of the wave with continuous distribution of frequencies and wave lengths means that the spectrum of combinative tones is also continuous. The data from the experiments where several isolated harmonics were superimposed /1/**(**Kazanov Yu.S., Kozlov V.V. and Levchenko V.Ya., Experiments on non-linear wave interaction in a boundary layer. Preprint 16, Novosibirsk, In-t teoret. i prikl. mekhaniki, SO AN SSSR, 1978.) show that when the oscillation amplitude increases and the non-linear stage of the process is reached, it is the amplitude of the combination tones that grows most rapidly. The amplitude of the fundamental harmonics grows more slowly. This leads one to the conclusion that the transition from laminar to turbulent flow of the fluid must occur within the wave packet very violently. Indeed, the measurements in /2/ show that the non-linear amplification of the originally monochromatic Tollmien-Schlichting (TS) wave from the unstable frequency range accompanied by the appearance of turbulent pulsations, lasts much longer than the explosive collapse of the wave packet and its transformation into a turbulent spot.

It is very propable that turbulent spots develop from the wave packets at the end of the non-linear stage of the laminar motion /3, 4/. This assumption is reinforced by the definite resemblance, mentioned in /5/, between the isolated spot in laminar flow and a laminar wave packet investigated in /6/. The pulsations occurring within the spot in the frequency range inherent in the selfexciting TS waves strenghten this resemblance.

1. Equations and boundary conditions.

In order to simplify the mathematical analysis of the wave packets, we shall assume that the Reynolds number $R \to \infty$. Then the initial Navier-Stokes equations will be reduced asymptotically to the simpler Prandtl equations, with selfinduced pressure remaining to be determined /7-9/. In connection with the three-

**Prikl.Matem.Mekhan.*,51,5,820-828,1987